The Poincaré metric on Bloch domains

Martin Chuaqui

University of Pennsylvania

Abstract

Using Ahlfors' form of Schwarz's lemma we obtain lower bounds for the density function of the Poincaré metric on Bloch domains in terms of the distance to the boundary.

Let Ω be a domain in the plane covered by the unit disc D. Thus Ω carries a complete metric $\lambda_\Omega(\zeta)|d\zeta|$ of constant curvature -1 given by $\lambda_\Omega(\pi(z))|\pi'(z)| = 2(1-|z|^2)^{-1}$, where $\pi : D \to \Omega$ is the covering map. In this short note, we will establish lower bounds for the Poincaré density λ_Ω by various powers of the distance to the boundary $\partial\Omega$, in the case when Ω is a Bloch domain. These are the domains not containing arbitrarily large discs, i.e., those for which

$$R_\Omega = \sup\{r > 0 : \Omega \text{ contains a disc of radius } r\} < \infty\,.$$

In an unpublished note, Beardon has shown that Bloch domains can also be characterized by the fact that for any $\alpha \in [0,1)$,

$$\inf_{\zeta\in\Omega} \lambda_\Omega(\zeta) d_\Omega^\alpha(\zeta) > 0\,, \quad \text{where} \quad d_\Omega(\zeta) = \operatorname{dist}(\zeta, \partial\Omega)\,.$$

Following Ahlfors' use of ultrahyperbolic metrics in [2] we prove

Theorem 1: *Let Ω be a Bloch domain. Then for any $\alpha \in (0,1)$,*

$$\lambda_\Omega d_\Omega^{1-\alpha} \geq c(\alpha,\Omega) > 0\,, \quad \textit{where} \quad c(\alpha,\Omega) = 2\alpha\left(\frac{1-\alpha}{1+\alpha}\right)^{1/2} R_\Omega^{-\alpha}.$$

Ahlfors' method was used originally to establish a lower bound for the size of unramified discs in the image of analytic functions $f : D \to C$ normalized by $|f'(0)| = 1$. He showed the estimate $\sqrt{3}/4$, and with basically the same argument one can prove the bound $1/2$ when f so normalized is locally schlicht. These bounds are not sharp, see [6] for further references. In general it is not true that $\inf \lambda_\Omega d_\Omega > 0$ as one can see by taking $\Omega = D^*$, the punctured disc. Domains for which this holds have been characterized by Beardon and Pommerenke in [3] and [7]. They introduce the function

$$\beta_\Omega(\zeta) = \inf\left\{\left|\log|\frac{\zeta - a}{b-a}|\right| : a,b \in \partial\Omega, |\zeta - a| = d_\Omega(\zeta)\right\}\,,$$

and show that $\lambda_\Omega d_\Omega \geq \dfrac{1}{2\sqrt{2}(k+\beta_\Omega)}$, with $k = 4+\log(3+2\sqrt{2})$. They deduce the corollary that $\lambda_\Omega d_\Omega \geq c > 0$ for some constant c iff there exists a constant $m < \infty$ such that every ring domain in Ω separating a boundary component of Ω has modulus $\leq m$. On the other hand, on an arbitrary domain Ω covered by D, the inequality $\lambda_\Omega d_\Omega \geq c > 0$ would follow from the generalized form of Schwarz's lemma if one could show that the metric $d_\Omega^{-1}(\zeta)|d\zeta|$ has curvature $\leq -c^2$ (even though d_Ω is in general not differentiable one can still define curvature in the distributional sense, see [4]). But this approach can not give back the result of Beardon and Pommerenke as is shown by taking, for example, Ω the disc minus a slit, say from -1/2 to 1/2. Then, at points interior to the region of points closest to either -1/2 or 1/2, $d_\Omega^{-1}(\zeta)|d\zeta|$ is flat. For a C^2 domain Ω, it can be shown nevertheless that the curvature of $d_\Omega^{-1}|d\zeta|$ is strictly negative. In fact, at points $\zeta \in \Omega$ where $d_\Omega(\zeta)$ is not attained uniquely on $\partial\Omega$, the generalized curvature of the metric is $-\infty$ [5]. Elsewhere, its curvature is $-(1-\kappa d_\Omega)^{-1}$, where κ is the geodesic curvature of $\partial\Omega$ at the closest point (this, with the sign convention so that $\kappa > 0$ if $\partial\Omega$ is convex at that point). This formula includes the cases when $\kappa d_\Omega = 1$, i.e., when the radius of curvature at $\partial\Omega$ equals d_Ω, since then the curvature of $d_\Omega^{-1}(\zeta)|d\zeta|$ also equals $-\infty$. Near an isolated boundary point of Ω, $\beta_\Omega = -\log d_\Omega$ and thus the result of Beardon and Pommerenke reads as

$$\lambda_\Omega d_\Omega \geq \frac{1}{2\sqrt{2}(k - \log d_\Omega)} .$$

Such an inequality can be established for Bloch domains in the same spirit as our first result:

Theorem 2: *If Ω is a Bloch domain then*

$$\lambda_\Omega d_\Omega \geq \frac{1}{1 + \log R_\Omega - \log d_\Omega} .$$

Proofs: The Poincaré density λ_Ω has the alternative definition

$$\lambda_\Omega(\zeta) = \inf\left\{ |f'(0)|^{-1} \Big| f : D \to \Omega \text{ is analytic and } f(0) = \zeta \right\} .$$

Since Ω is covered by the unit disc D, an extremal map for λ_Ω is of the form $f = \pi \circ F$, where F is a Möbius selfmap of D. In particular, f is locally schlicht. For $\alpha \in (0,1)$ we consider the conformal metric $\rho(\zeta)|d\zeta|$ in Ω given by

$$\rho(\zeta) = \frac{2\alpha A}{R(\zeta)^{1-\alpha}(A^2 - R(\zeta)^{2\alpha})} ,$$

where A is a constant and $R(\zeta)$ is the radius of the largest unramified disc about ζ in the image of f. It follows that $R(\zeta) = d_\Omega(\zeta)$. As in [2], we will show that for A sufficiently large, the metric $\rho(f(z))|f'(z)||dz|$ is ultrahyperbolic in D. At $z_0 \in D$, a supporting metric is given by $\rho_0(z)|dz|$ with

$$\rho_0(z) = \frac{2\alpha A|f'(z)|}{|f(z) - a|^{1-\alpha}(A^2 - |f(z) - a|^{2\alpha})} ,$$

where $a \in \partial\Omega$ is such that $|f(z_0) - a| = d_\Omega(f(z_0))$. It is easy to check that near z_0, $\rho_0(z)|dz|$ has curvature -1, and since $R(f(z)) \leq |f(z) - a|$, in order to have $\rho_0(z) \leq \rho(f(z))|f'(z)|$ we need the function $x^{1-\alpha}(A^2 - x^{2\alpha})$ to be increasing. This will be the case provided $A^2 >$

$(1+\alpha)(1-\alpha)^{-1}R_\Omega^{2\alpha}$. Hence, for such choices of A, $\rho(f(z))|f'(z)||dz|$ is ultrahyperbolic, and therefore

$$\rho(f(z))|f'(z)| \le 2(1-|z|^2)^{-1}\,.$$

At $z=0$ we obtain

$$\alpha A|f'(0)| \le d_\Omega^{1-\alpha}(\zeta)(A^2 - d_\Omega^{2\alpha}(\zeta))\,, \quad \text{where} \quad \zeta = f(0)\,. \qquad (*)$$

We will derive our results from this inequality. First, $\alpha A|f'(0)| \le A^2 d_\Omega^{1-\alpha}(\zeta)$, thus

$$\lambda_\Omega(\zeta) = 2|f'(0)|^{-1} \ge \frac{2\alpha}{A d_\Omega^{1-\alpha}(\zeta)}\,,$$

and Theorem 1 follows by letting A tend to $(1+\alpha)^{1/2}(1-\alpha)^{-1/2}R_\Omega^\alpha$.
As an example we consider $\Omega = D^*$. Then $\lambda_\Omega(\zeta) = -\frac{1}{|\zeta|\log|\zeta|}$ and $\inf \lambda_\Omega d_\Omega^{1-\alpha}$ is attained when $-\log|\zeta| = \alpha^{-1}$. Hence, $\lambda_\Omega d_\Omega^{1-\alpha} \ge e\alpha$, which shows that $c(\alpha,\Omega)$ has the right behavior for small α.

Finally, to prove Theorem 2, we let A in $(*)$ tend to its critical value. This yields

$$d_\Omega(\zeta)|f'(0)|^{-1} \ge \left(\frac{1+\alpha}{1-\alpha}\right)^{1/2}\left(\frac{R_\Omega}{d_\Omega(\zeta)}\right)^\alpha \frac{\alpha}{\left(\frac{1+\alpha}{1-\alpha}\right)\left(\frac{R_\Omega}{d_\Omega(\zeta)}\right)^{2\alpha} - 1}\,,$$

and the theorem follows by letting $\alpha \to 0$.

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